METHODS FOR IDENTIFICATION OF
DYNAMIC PARAMETERS OF MATHEMATICAL -
STRUCTURAL MODELS FROM EXPERIMENTAL DATA

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ABSTRACT

The dynamic analysis of complex and critical structures, such as nuclear power plants, must not only employ both theoretical and experimental analysis, but also combine these two approaches to provide a unified description of the structure. A theoretical *a priori* model is useful in planning a vibration test, while the data obtained from such testing can be used to modify the *a priori* model so that it reproduces the experimental data.

This paper illustrates several mathematical methods useful in planning the placement of excitors and accelerometers; analyzing data to identify resonant frequencies, modal damping and mode shapes; modifying the mass and stiffness properties of the *a priori* model (and consequently indicate improved methods of modeling in future studies); and identifying unknown forces acting on the structure, once a good model has been obtained.

Central to these methods is the use of a generalization of the inverse of a matrix - the pseudo-inverse and the availability of efficient computer software for its computation.

The properties of the pseudo-inverse allow it to calculate weighted least-mean-square and weighted minimum-norm solutions to ill-conditioned linear systems. It is consequently a versatile tool in the analysis of vibrating systems where the number of measurement and forcing points are often different and the confidence in different measurements and models varies.

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METHODS FOR IDENTIFICATION OF DYNAMIC PARAMETERS OF MATHEMATICAL STRUCTURAL MODELS FROM EXPERIMENTAL DATA

P. IBÁÑEZ

1. Introduction

The reduction of a complex structure to a mathematical model results from a blending of skill, experience, and good judgement. The elegant mathematical end product must not obscure the roots of the model in such subjective disciplines. Similarly, rather than treating a structure like a black box, the vibration testing of a complex structure should be planned with reference to an "a priori model" of the structure. This model must be capable of directing the initial phases of testing and in turn of being modified by the experimental results to yield a "modified model" which reproduces experimental data. Consequently, the dynamic analysis of complex and critical structures must not only employ both theoretical and experimental analysis but also combine these two approaches to provide a unified description of the structure. This paper presents several examples of methods for the combined use of theoretical and experimental analysis.

These methods are intended as examples and suggestions rather than a state-of-the-art review, although they have proven themselves successful in application to real structures. The reader may wish to refer to other works done in the fields of vibration testing and parameter identification. Smith and his associates [1,2], for example, have carried out extensive testing of nuclear power plants. Parameter identification of vibrating structures was the topic of a recent symposium, edited by Pilkey and Cohen [3], as well as an earlier summary by Young and On [4]. Especially noteworthy are the works of Collins and his associates [5,6].

2. A Mathematical Preliminary--The Pseudo-Inverse

The pseudo-inverse is a generalization of the inverse of a matrix. It has allowed a generalization and extension of my earlier work (see Ibáñez [7]), is related to but more versatile than methods used by Collins [5,6] and has been used in the analysis of geological seismic data by Jackson [8]. A detailed discussion of the properties and computation of the pseudo-inverse are found in reference [8] and in Larson's work, [9].

Consider a set of linear equations

$$AX = Y \tag{1}$$

where A is an n x m matrix, X is an m x ℓ column vector and Y is an n x ℓ column vector. Problems in model, parameter and force identification often reduce to the solution of eq. (1) when n is not equal to m, in which case the classical solution

$$X = A^{-1}Y \tag{2}$$

is not valid as A^{-1} does not exist.

The pseudo-inverse of a matrix A, however, denoted by A^I, always exists, regardless of the values of m and n and the linear dependence of the columns and rows of A. Thus, the "solution"

$$Z = A^I Y \tag{3}$$

can always be defined. I present, without proof, several properties of A^I and Z:

- If A^{-1} exists then $A^I = A^{-1}$ and

$$AZ = Y \tag{4}$$

(i.e., the pseudo-inverse equals the classical inverse).

- If m < n (more equations than unknowns) or A is otherwise overdetermining X, then Z is the least mean square solution to eq. (1). That is,

$$(AZ - Y)^T(AZ - Y) \leq (AX - Y)^T(AX - Y) \tag{5}$$

for any X.

- If m > n (fewer equations than unknowns) or A is otherwise undetermining X, then of all possible solutions, X, Z has the minimum norm. That is,

$$Z^TZ \leq X^TX \tag{6}$$

for any X.

Two additional properties of the pseudo-inverse extend its use. Often the elements of the vector Y vary greatly in size or the uncertainties in the individual elements differ. We can define a vector W as

$$\sigma W = Y \tag{7}$$

where σ is a weighting matrix. For example, σ may be a diagonal matrix of standard deviations (square root of variance) of the elements of Y. In any case, the W elements are then of roughly equal size or variance. Substitution of eq. (7) into eq. (1) yields

$$\sigma^{-1}AX = W \tag{8}$$

which is solved by

$$X \simeq Z = [\sigma^{-1}A]^IW = [\sigma^{-1}A]^I\sigma^{-1}Y \tag{9}$$

The least-mean-square property of the pseudo-inverse solution, eq. (5), now becomes

$$(AZ - Y)^T\sigma^{-T}\sigma^{-1}(AZ - Y) \leq (AX - Y)^T\sigma^{-T}\sigma^{-1}(AX - Y) \tag{10}$$

Thus, a weighted least-mean-square solution results.

Similarly, the solution X may be weighted by the substitution

$$\delta T = X$$

to yield

$$X \simeq Z = \delta[A\delta]^IY \tag{11}$$

and the minimum norm property (eq. (6)) becomes

$$Z^T\delta^T\delta Z \leq X^T\delta\delta^TX \tag{12}$$

Both types of weighting can be combined in the solution

$$X \simeq \delta[\sigma^{-1}A\delta]^I\sigma^{-1}Y \tag{13}$$

The second property of interest requires more insight into the construction of the pseudo-inverse. Any matrix A can be expressed as its "singular value decomposition"

$$A = USV^T \tag{14}$$

where U is an n x n set of orthogonal vectors ($U^TU = I$), V is an m x m set of orthogonal vectors ($V^TV = I$) and S is an n x m matrix containing zeros everywhere except on its principal diagonal. On this diagonal, S contains the "singular values" of A, some of which may be zero.

The pseudo-inverse of A is given by

$$A^I = VS^*U^T \tag{15}$$

where S* is the m x n matrix formed by transposing S and replacing all nonzero terms by their inverses.

If the ratio of the largest to smallest nonzero singular value is so large that the smaller one might as well be considered zero, the matrix A is "almost singular" and has a large "condition number." This indicates that the information in eq. (1) using A is not as complete as might first appear--some equations are, for all intents and purposes, linearly dependent. The development of S* in eq. (15) allows us to set to zero any exceedingly small singular value and explicitly recognize this in the development of A^I. This avoids numerical problems as the inverse of a small number is very large and would, falsely, dominate A^I if not excluded.

3. Use of the A Priori Model to Direct Vibration Testing

Two problems faced by the test engineer are the proper placement of vibrators and accelerometers. I shall be concerned here only with forced vibration tests using sinusoidal vibrators, although other forms of tests can be approached in a similar manner. The object of a vibration test is to adequately excite and measure the important modes of vibration of the structure. The data analysis is easier if, one mode at a time, each mode is excited and detected significantly more than all others.

Consider first the case of testing a structure with n modes using m vibrators. The effective force exciting each mode is given by

$$\begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{matrix} = E = \phi^T F = \phi^T \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{matrix} \tag{16}$$

where e_i = effective force on mode i,
f_ℓ = force on location ℓ,
$[\phi]_{\ell i}$ = mode shape at location ℓ for mode i.

The maximum acceleration response due to a mode occurs roughly at its resonant frequency and is given by

$$[\ddot{y}_i]_{max} \approx \frac{e_i}{2\beta_i W_i} \tag{17}$$

where β_i = modal damping for mode i,
W_i = effective mass for mode i.

For frequencies other than the resonant frequency, eq. (17) represents an upper limit on the modal response. Suppose that we wish to excite mode j strongly and suppress all other modes. Thus, we set

$$[\ddot{y}_i]_{max} = \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix} \tag{18}$$

Combining eq. (16) through eq. (18) yields

$$\begin{matrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{matrix} = \begin{matrix} 1/2\beta_i W_i & & 0 \\ & \ddots & \\ 0 & & 1/2\beta_n W_n \end{matrix} \quad \phi^T F_j \equiv B\phi^T F_j \tag{19}$$

where F_j is the force to excite the j-th mode. This equation may be solved by the pseudo-inverse method for the required forcing

$$F_j = [B\ {}^T]^I \begin{matrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{matrix} \tag{20}$$

The forces needed for the dominant excitation of each mode in turn is given by the matrix

$$\tilde{F} = [F_1, F_2, \ldots F_n] = [B\phi^T]^I * (n \times n \text{ identity matrix}) \tag{21}$$
$$[B\phi^T]^I$$

Thus, the pseudo-inverse of the weighted shape matrix has as columns, the force distribution required to, in turn, excite each mode significantly more than the others.

One naturally asks how well these forces can selectively excite each mode. Let the matrix $\tilde{Y}$ be the response from forcing by $\tilde{F}$. From eq. (19) and eq. (21) there results

$$\tilde{Y} = B\phi^T\tilde{F} = [B\phi^T][B\phi^T]^I \tag{22}$$

Clearly, if $\tilde{Y}$ is close to the identity matrix, we have found a very good way to individually excite each mode.

As an example, consider the three-story structure shown in Figure 1, with additional properties given in Table I. This structure was obtained from Biggs' book [10] on structural dynamics. Consider testing the Biggs' Inflexible Girder Structure (BIGS) using one, two or three vibrators to define the three modes of vibration. Thus, we must form $\tilde{F} = [B\phi^T]^I$ and $\tilde{Y} = [B\phi^T][B\phi^T]^I$ for seven cases as indicated in Table II.

This table lends itself to many interpretations which coincide with the test engineer's intuition. Consider the case of one vibrator on floor one. The $\tilde{Y}$ matrix is not close to an identity matrix--hence indicating, quite correctly, that it would be difficult to produce "pure response" with only one vibrator on floor one. Similar observations hold for single vibrators on floors two or three. Nevertheless, these results indicate that forcing on floor two is the best for isolation of mode one; forcing on floor one is best for isolation of mode two and three.

If two vibrators are used (cases 4-6), $\tilde{Y}$ matrices result which are closer to identity matrices, indicating a greater ability to selectively excite modes, as expected. From these results, an engineer with only two vibrators would probably place them on floors 2 and 3, as this would allow a fair selection of each of all three modes. If his interests lie with the second mode only, for example, he might instead place them on floors 1 and 3. In any case, the $\tilde{F}$ matrix gives the optimal force distribution of the two vibrators. Lastly, if three vibrators are available, a perfect selection of modes is possible, as indicated in case 7 by $\tilde{Y} = I$. This result is anticipated, of course. In addition, the desired force distribution is given by $\tilde{F}$.

The second problem faced by the test engineer is the placement of accelerometers and analysis of their response. A sufficient number of accelerometers must be correctly placed on the structure to adequately define each mode of vibration--its resonant peak and its shape (eigenvector). The response measured by the accelerometers is given by

$$\ddot{X} = \phi\ddot{Y} \tag{23}$$

where x_j is the response of the accelerometer at location j. Note that the shape matrix ϕ used here may be defined at many more or different points than that used in dealing with vibrators. Thus, if we are measuring at ℓ locations and there are n modes, ϕ is ℓ x n. As we seek to isolate modal response in our data analysis, it is natural to use the pseudo-inverse to yield an estimate of $\ddot{Y}$ based upon the measured $\ddot{X}$

$$\ddot{Y} = \phi^I\ddot{X} \tag{24}$$

Should some of the accelerometer responses be considered more accurate than others (due to instrument sensitivities, noise, etc.), a weighting can be placed on each channel as was discussed in Section 2.

$$\ddot{Y} = [\sigma^{-1}\phi]^I\sigma^{-1}\ddot{X} \tag{25}$$

For simplicity, I will proceed with eq. (24).

One naturally asks if the estimator (eq. (24)) is a good one. Substitution of eq. (23) into eq. (24) yields

$$\ddot{Y} \approx \phi^I\phi\ddot{Y} \tag{26}$$

If $\phi^I\phi$ is very close to the identity matrix, then the placement of accelerometers allows a good definition of each mode. For example, consider measuring the response of the BIGS with one, two, or three accelerometers while looking for the 1-st and 2-nd modes only. This yields seven possible accelerometer placement schemes as indicated in Table III.

Clearly, the first three cases, using only one accelerometer, cannot yield good estimates of both modes. Thus, for example, one accelerometer on floor one "sees" both modes on an equal basis. Note that one accelerometer on floor 2 is effective in seeing mode 1 above mode 2. As anticipated, two accelerometers, regardless of floor can distinguish between the two modes (remember that we assume no third mode is present). However, inspection of the ϕ^I matrix indicates that configuration 5 uses the two accelerometers more evenly than the other two, and might thus be preferable.

Lastly, consider measuring on all three floors. The two modes are now overdetermined, but the ϕ^I matrix estimator indicates how they must be averaged to resolve the modes. That this is the proper way is indicated by the result $\phi^I\phi = I$.

The two methods just discussed to aid in placement of excitors and accelerometers must be used in conjunction with, and not as a substitute for, good sense. Clearly, not all locations on a structure are suitable for locating vibrators and accelerometers. Even so, the number of permutations (only seven for the BIGS) will grow astronomically for more complex structures. The engineer must use his judgement to select a few potential configurations and then check for their suitability by the methods thus presented. Also, never forget that all these calculations are based on the a priori model which is not entirely correct. Indeed, the vibration test is carried out because we are unsure of the structure (which is probably full of surprises).

4. Identification of Eigenparameters

The a priori model will not perfectly predict the results of the experimental vibration test. This will result in a discrepancy in "eigenparameters"--the eigenvectors (mode shapes), the eigenvalues (resonant frequencies), the effective modal masses and the effective modal dampings.

The transformation of the data in eq. (24), $\ddot{Y} = \phi^I \ddot{X}$, has a great advantage in that it produces n functions as opposed to the original m functions. In many applications m >> n, that is, the number of transducers greatly exceeds the number of significant modes and this transformation consequently reduces the amount of information that must be processed.

Of course, this transformation does not produce purely modal response as the a priori ϕ is not exact. Nevertheless, we can now address ourselves to identification of an n-degree-of-freedom system

$$M\ddot{Y} + C\dot{Y} + KY = E \tag{27}$$

where the n x n mass, damping, and stiffness matrices--M, C, and K--are almost diagonal. We thus look for n x n matrices Ψ, Ω, β, and W such that the transformation

$$Y = \Psi Z \tag{28}$$

yields

$$\begin{aligned} Z + 2\beta\Omega\dot{Z} + \Omega^2 Z &= \Psi^T E \\ \Psi^T M \Psi &= W \\ \Psi^T C \Psi &= W(2\beta\Omega) \\ \Psi^T K \Psi &= W(\Omega^2) \end{aligned} \tag{29}$$

where Ω, β, and W are diagonal matrices of eigenfrequencies, modal dampings, and effective masses of both eq. (27) and the tested structure. Then, the mode shapes of the structure, ϕ_e, as modified by the experimental data are given by

$$\phi_e = \phi\Psi \tag{30}$$

where ϕ is the a priori mode shape matrix. Thus, Ψ is a correction to the a priori model as supplied by the experimental data.

A method for finding Ψ and the associated Ω, β, and W was presented by Ibáñez [7] and has since been improved upon (see Ibáñez [11]). Identification of these parameters proceeds through the minimization of the error between experimental and model response, as a function of the unknown parameters. The error is defined by a criterion function, and its minimization carried out by gradient techniques. This minimization is greatly facilitated by reducing the number of unknown parameters from m x n in ϕ to only n x n in Ψ.

As an example, consider Biggs' Inflexible Girder Structure (BIGS) and Biggs' Flexible Girder Structure (BFGS) (whose properties are given in Figure 2 and Table IV). Note that these two models come from the same hypothetical structure except that in the first case the floor girders are assumed rigid, while in the second, their flexibility is taken into account.

Postulate that a design engineer has analyzed Biggs' structure under the rigid girder assumption and produced an a priori model--BIGS. The assumption is symbolic of the assumptions made in reducing any complex structure to a mathematical model. The test engineer, however, performs a test on the actual structure and consequently his data is effected by the flexibility of the girders. A hypothetical forced vibration test with a vibrator on floor 1 of Biggs' structure using the BFGS model yields the response for each floor as shown in Figure 3.

Given this "experimental" response, our task is to modify the a priori BIGS model to yield the BFGS model. To make the task more realistic, the a priori (BIGS) damping were assumed to be 0.14, 0.07, and 0.12% of critical, thus requiring an improvement in β as well as Ω and W. Using the procedures discussed in this section a Ψ matrix was found and as well as Ω_e, β_e, and W_e matrices. These, and the modified models shape matrix, $\phi_e = \phi_a\Psi$, is shown in Table V. As can be seen, it has been possible to approximate the correct BFGS eigenparameters from the "experimental" data. While the average difference is eignevalues between the BIGS and BFGS model is 7.00%, the identified eigenvalues are within 0.08% of the BFGS model. The corresponding figures for damping are 30.0% and 0.0%; for mode shape 21.4% and 1.4%.

5. Modification of the A Priori Model

From the identification scheme, we have a new set of eigenparameters which can be expressed as perturbations on the a priori model eigenparameters:

$$\phi_e = \phi_a\Psi \equiv \phi_a(I + \Psi') \quad (31)$$

$$\Omega_e = \Omega_a(I + D) \quad (32)$$

$$W_e = W_a(I + G) \quad (33)$$

We now compare the differences between the a priori and experimental models mass and stiffness matrices. The values of Ψ', D, and G will be used to modify the mass (M) and stiffness (K) matrices of the a priori model so that it reproduces the experimental data.

First, note that Ψ', D, and G do not contain sufficient information to uniquely define the required changes in M and K. The latter contain only n^2 "knows" (typically the number of significant modes--$n < 10$) while the former contain m^2 "unknowns" (typically the number of locations--$m > 100$). Secondly while any M and K reproducing the experimental results would be useful, we are ultimately interested in finding out which portion of the a priori model was incorrect--and why. Consideration of these two points requires the introduction of additional information. In particular, the engineer must specify his confidence in the various portions of the a priori model, and limit the ways in which the various portions may vary to achieve the modified M and K.

Let M_p and K_p be the p-th way in which the mass and stiffness is allowed to vary away from the a priori model. Let σ_p be a standard deviation, giving the engineer's judgement as to how likely the p-th way of variation is. For example, an engineer may use a small value of σ_p if the p-th way of variation is due to material properties such as Young's modulus, or density, in which he has confidence, while he may use a large value if the p-th way of variation is due to effective joint stiffness, which is uncertain.

The total change in stiffness is a weighted sum of the allowable changes

$$M_{total} = \sum_p \alpha_p M_p \tag{34}$$

$$K_{total} = \sum_p \alpha_p K_p \tag{35}$$

Similarly, the total change in eigenparameters--eigenvalues, for example--is

$$D_{total} = \sum_p \alpha_p D_p \tag{36}$$

where D_p is the change due to variation M_p and K_p. Note that we have assumed that $\alpha_p M_p$ and $\alpha_p K_p$ are sufficiently small so that linear perturbation theory applies.

Clearly, a desirable relation is that the total change in eq. (36) equals the identified change in eq. (32):

$$D = \sum_p \alpha_p D_p \tag{37}$$

Our task is to solve for a set of α_p such that eq. (37) is satisfied as closely as possible. These α_p will then give the difference between the a priori model and the experimental structure. Also, by referring to what part of the model corresponds to the p-th way of variation, the engineer can determine where and why the a priori model was incorrect.

Collins [5,6] has developed a program which looks explicitly at the sensitivity of the eigenparameters with respect to the input parameters of a finite element model. His sensitivities are analogous to D_p. He then develops a scheme to solve, essentially, for the α_p weighted according to the engineer supplied uncertainties, σ_p. This program requires detailed knowledge and manipulation of the finite element modeling program, but is all the more powerful and efficient due to this explicit procedure.

In a separate approach (Ibáñez [11]), I use a general finite element program to implicitly determine the sensitivities. The finite element program is run p times, in each case with a perturbation in the p-th way of variation. Comparing this with an unperturbed run yields the mass and stiffness perturbations M_p and K_p. Using perturbation theory, the value of D_p can then be quickly calculated. This approach is simpler to program and can be used in conjunction with any structural analysis program. These gains, however, are offset by the longer running time required by repeated running of the program. Nevertheless, this approach also allows weighting by the σ_p, which is carried out by using the pseudo-inverse (see Ibáñez [11]).

As an example, consider the Biggs' BIGS/BFGS problem using the identified values of D_p from Section 4. Suppose that six allowable modes of change are used, corresponding to the variation of three inter-floor springs and three floor-to-ground springs. The allowable K_p are shown in Table VI as well as the modified stiffness. As can be seen in Table VI, the correct K matrix has been approximated. While the BIGS and BFGS K matrices differ by 15%, the identified and BFGS matrices differ by only 5%.

6. Identification of Operational Forces

One use of the modified model is to identify forces acting on a structure. Thus, for example, assume that the structure is being acted upon by unknown transient forces at m points, and that we are measuring its response at ℓ points. If n modes are being significantly excited then the modal response is given by

$$\ddot{Y} = \phi_1^I \ddot{X} \tag{38}$$

where ϕ_1 is an ℓ x n matrix and $\ddot{X}$ is the measured acceleration. The modal excitation is given by

$$E = \phi_2^T F \tag{39}$$

where ϕ_2 is an m x n matrix. (Both ϕ_1 and ϕ_2 are appropriately selected portions of the modified model shape matrix.) From eq. (39) we have

$$F = [\phi_2^T]^I E \tag{40}$$

Now

$$\mathcal{F}[\ddot{Y}]_i = h_i(\omega)\mathcal{F}[E]_i \tag{41}$$

where $\mathcal{F}$ denotes the Fourier transform and h_i is the modal transfer function for the i-th mode.

$$h_i(\omega) = \frac{-\left(\frac{\omega}{\omega_i}\right)^2}{1 - \left(\frac{\omega}{\omega_i}\right)^2 + 2\sqrt{-1}\,\beta_i \frac{\omega}{\omega_i}} \tag{42}$$

Thus

$$\mathcal{F}(E) = \begin{bmatrix} 1/h_i & & & 0 \\ & 1/h_2 & & \\ & & \ddots & \\ 0 & & & 1/h_n \end{bmatrix} \mathcal{F}(\ddot{Y}) \equiv H^{-1}\mathcal{F}(\ddot{Y}) \tag{43}$$

and combining eq. (38), eq. (40), and eq. (43) yields

$$\mathcal{F}(F) = [\phi_2^T]^I H^{-1}\mathcal{F}([\phi_1]^I \ddot{X}) \tag{44}$$

The use of eq. (44) and the fast Fourier transform (Cooley-Tukey) algorithm allows us to compute F or its Fourier transform, quite easily:

$$F = [\phi_2^T]^I \mathcal{F}^{-1}(H^{-1}\mathcal{F}([\phi_1]^I \ddot{X})) \tag{45}$$

These expressions minimize the number of Fourier transforms taken to 3 x n, where n is the number of modes significantly excited.

As an example, Figure 4 shows the transient forces used to excite the BFGS on floors 1 and 2, and the resulting response. The arrows indicate the various steps in evaluating eq. (45). The identified forces are compared to the actual forces in Figure 5. As can be seen, a close identification of the forces has been achieved.

This method can be used even when the number and location of exciting forces is not the same as the measurement locations, or regardless of the number of modes involved, (although the modes must cover a frequency range comparable to the frequency range of the input forces). As before, selective weighting of each accelerometer is possible.

7. Inferences

This paper has illustrated several mathematical methods useful in the planning and analysis of vibration tests. I have shown how the pseudo-inverse can perform weighted least-mean-square and minimum norm solutions of ill-conditioned linear systems, and consequently is of great use in placing vibrators and accelerometers, analyzing data to identify eigenparameters, adjusting mass and stiffness matrices to fit experimental data, and identifying unknown forces acting on a known structure.

Future work to integrate theoretical and experimental dynamic analysis should be concerned with improving these methods by using them in a variety of actual vibration tests. In this way, the optimal algorithms can be determined. These methods could also benefit from a more rigorous mathematical development dealing with the effects of noise, error, and uncertainty in the data. Nonlinear effects in structures must be considered. As a minimum, all vibration tests should be carried out at several different levels of forcing and response and an appropriate linear model identified at each level. Differences between these models will point to important nonlinear effects.

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TABLE I

BIGGS' INFLEXIBLE GIRDER STRUCTURE--BIGS

	24,630	0	0	
M =	0	23,133	0	kilograms
	0	0	11,567	
	13.17	-7.787	0	
K =	-7.787	15.570	-7.787	x 10^6 newtons/meter
	0	7.787	7.787	
	1.000	1.000	1.000	
ϕ =	1.471	-0.146	-2.220	
	1.639	-1.041	2.680	
Mode:	1	2	3	
ω_o	8.325	24.06	35.09	(radians/sec)
F_o	1.325	3.830	5.584	(Hertz)
β	0.100	0.100	0.100	(fraction of critical)
W	105,800.	37,660.	221,700.	(kilogram)

TABLE II

DETERMINATION OF OPTIMAL VIBRATOR PLACEMENT

$$B = \begin{bmatrix} 4.726 & 0 & 0 \\ 0 & 13.277 & 0 \\ 0 & 0 & 2.255 \end{bmatrix} \times 10^{-5}\ (\text{kilogram})^{-1}$$

Comment	ϕ^T			$B\phi^T \times 10^5$			$[B\phi^T]^I \times 10^{-5}$			$[B\phi^T][B\phi^T]^I$		
1. 1 vibrator	1.00			4.73			0.0232	0.0652	0.0111	0.111	0.308	0.052
on floor 1	1.00			13.28						0.308	0.865	0.147
	1.00			2.26						0.052	0.147	0.025
2. 1 vibrator	1.47			6.95			0.0901	-0.0251	-0.0649	0.626	-0.174	-0.451
on floor 2	-0.15			-1.94						-0.174	0.049	0.126
	-2.22			-5.01						-0.451	0.126	0.325
3. 1 vibrator	1.64			7.75			0.0269	-0.0481	0.0210	0.209	-0.372	0.163
on floor 3	-1.04			-13.82						-0.372	0.664	-0.291
	2.68			6.04						0.163	-0.291	0.127
4. 2 vibrators	1.00	1.47		4.73	6.95		0.0251	0.0647	0.0098	0.754	0.156	-0.401
one on floor 1	1.00	-0.15		13.28	-1.96		0.0914	-0.0216	-0.0644	0.156	0.901	0.254
one on floor 2	1.00	-1.04		2.26	-5.01					-0.401	0.254	0.344
5. 2 vibrators	1.00	1.64		4.73	7.75		0.0586	0.0484	0.0356	0.696	0.030	0.459
one on floor 1	1.00	-1.04		13.28	-13.28		0.0541	-0.0256	0.0375	0.630	0.997	-0.046
one on floor 3	1.00	2.68		2.26	6.04					0.459	0.046	0.307
6. 2 vibrators	1.47	1.64		6.95	7.75		0.0819	0.007	-0.089	0.669	-0.333	-0.334
one on floor 2	-0.15	-1.04		-1.96	-13.82		0.0126	-0.049	-0.037	-0.333	0.688	-0.333
one on floor 3	-2.22	2.68		-5.01	6.04					-0.334	-0.333	0.665
7. 3 vibrators	1.00	1.47	1.64	4.73	6.95	7.75	0.0494	0.0493	0.0495	1.00	0.00	0.00
one on each	1.00	-0.15	-1.04	13.28	-1.94	-13.82	0.0680	-0.0068	-0.1027	0.00	1.00	0.00
floor	1.00	-2.22	2.68	2.26	-5.01	6.04	0.0379	-0.0240	0.0620	0.00	0.00	1.00

TABLE II

DETERMINATION OF OPTIMAL VIBRATOR PLACEMENT

$$B = \begin{bmatrix} 4.726 & 0 & 0 \\ 0 & 13.277 & 0 \\ 0 & 0 & 2.255 \end{bmatrix} \times 10^{-5} \text{ (kilogram)}^{-1}$$

Comment	ϕ^T			$B\phi^T \times 10^5$			$[B\phi^T]^I \times 10^{-5}$			$[B\phi^T][B\phi^T]^I$		
1. 1 vibrator	1.00			4.73			0.0232	0.0652	0.0111	0.111	0.308	0.052
on floor 1	1.00			13.28						0.308	0.865	0.147
	1.00			2.26						0.052	0.147	0.025
2. 1 vibrator	1.47			6.95			0.0901	-0.0251	-0.0649	0.626	-0.174	-0.451
on floor 2	-0.15			-1.94						-0.174	0.049	0.126
	-2.22			-5.01						-0.451	0.126	0.325
3. 1 vibrator	1.64			7.75			0.0269	-0.0481	0.0210	0.209	-0.372	0.163
on floor 3	-1.04			-13.82						-0.372	0.664	-0.291
	2.68			6.04						0.163	-0.291	0.127
4. 2 vibrators	1.00	1.47		4.73	6.95		0.0251	0.0647	0.0098	0.754	0.156	-0.401
one on floor 1	1.00	-0.15		13.28	-1.96		0.0914	-0.0216	-0.0644	0.156	0.901	0.254
one on floor 2	1.00	-1.04		2.26	-5.01					-0.401	0.254	0.344
5. 2 vibrators	1.00	1.64		4.73	7.75		0.0586	0.0484	0.0356	0.696	0.030	0.459
one on floor 1	1.00	-1.04		13.28	-13.28		0.0541	-0.0256	0.0375	0.630	0.997	-0.046
one on floor 3	1.00	2.68		2.26	6.04					0.459	0.046	0.307
6. 2 vibrators	1.47	1.64		6.95	7.75		0.0819	0.007	-0.089	0.669	-0.333	-0.334
one on floor 2	-0.15	-1.04		-1.96	-13.82		0.0126	-0.049	-0.037	-0.333	0.688	-0.333
one on floor 3	-2.22	2.68		-5.01	6.04					-0.334	-0.333	0.665
7. 3 vibrators	1.00	1.47	1.64	4.73	6.95	7.75	0.0494	0.0493	0.0495	1.00	0.00	0.00
one on each	1.00	-0.15	-1.04	13.28	-1.94	-13.82	0.0680	-0.0068	-0.1027	0.00	1.00	0.00
floor	1.00	-2.22	2.68	2.26	-5.01	6.04	0.0379	-0.0240	0.0620	0.00	0.00	1.00

TABLE III

DETERMINATION OF OPTIMAL ACCELEROMETER PLACEMENT
(for search of 1st and 2nd modes)

Comment	ϕ		ϕ^I	$\phi^I\phi$	
1. 1 accelerometer, on floor 1	1.000	1.000	0.500 0.500	0.500 0.500	0.500 0.500
2. 1 accelerometer, on floor 2	1.471	-0.146	0.673 -0.067	0.990 -0.098	-0.098 0.010
3. 1 accelerometer, on floor 3	1.639	-1.041	0.435 -0.276	0.713 -0.453	-0.453 0.286
4. 2 accelerometers, one on floor 1, one on floor 2	1.000 1.471	1.000 -0.146	0.090 0.618 0.910 -0.618	1.00 0.00	0.00 1.00
5. 2 accelerometers, one on floor 1, one on floor 3	1.000 1.639	1.000 -1.041	0.388 0.373 0.612 -0.373	1.00 0.00	0.00 1.00
6. 2 accelerometers, one on floor 2, one on floor 3	1.471 1.639	-0.146 -1.041	0.806 -0.113 1.269 -1.139	1.00 0.00	0.00 1.00
7. 3 accelerometers, one on each floor	1.000 1.471 1.639	1.000 -0.146 -1.041	0.264 0.258 0.217 0.581 0.044 -0.400	1.00 0.00	0.00 1.00

TABLE IV

BIGGS' FLEXIBLE GIRDER STRUCTURE--BFGS

$$M = \begin{matrix} 24{,}630 & 0 & 0 \\ 0 & 23{,}133 & 0 \\ 0 & 0 & 11{,}567 \end{matrix} \quad \text{kilograms}$$

$$K = \begin{matrix} 12.73 & -7.752 & 0.386 \\ -7.752 & 14.35 & -7.015 \\ 0.386 & -7.015 & 6.629 \end{matrix} \quad \times 10^6 \text{ newtons/meter}$$

$$\phi = \begin{matrix} 1.000 & 1.000 & 1.000 \\ 1.541 & -0.0515 & -1.873 \\ 1.758 & -1.123 & 2.075 \end{matrix}$$

Mode:	1	2	3	
ω_o	7.746	22.65	33.65	(radians/sec)
f_o	1.233	3.605	5.355	(Hertz)
β	0.100	0.100	0.100	(fraction of critical)
W	115,300.	39,280.	155,600.	(kilograms)

TABLE V

RESULT OF EIGENPARAMETER IDENTIFICATION

$$\Psi = \begin{matrix} 1.00000 & 0.01620 & 0.00071 \\ -0.03075 & 1.00000 & 0.15070 \\ 0.00153 & -0.03281 & 1.00000 \end{matrix}$$

(normalized to unity diagonal terms)

	A Priori Values (BIGS)			Exact Values (BFGS)			Identified Values		
Ω (Hertz)	1.325	3.830	5.584	1.233	3.605	5.355	1.233	3.610	5.360
β (fraction of critical)	0.140	0.070	0.120	0.100	0.100	0.100	0.100	0.100	0.100
W (kilograms)	105,800.	37,660.	221,700.	115,300.	39,780.	155,600.	115,300.	39,780.	155,600.
φ	1.000 1.471 1.639	1.000 -0.146 -1.041	1.000 -2.220 2.680	1.000 1.541 1.758	1.000 -0.0515 -1.123	1.000 -1.873 2.075	1.007 1.527 1.737	1.000 -0.0502 -1.121	0.981 -1.908 2.150

TABLE VI

IDENTIFICATION OF STIFFNESS MATRIX

Allowable Ways of Variation (newtons/meter):

$$K_1 = \begin{bmatrix} 1.0 & -1.0 & 0.0 \\ -1.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}, K_2 = \begin{bmatrix} 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 1.0 \end{bmatrix}, K_3 = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & -1.0 \\ 0.0 & -1.0 & 1.0 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}, K_5 = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}, K_6 = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

A Priori Stiffness (BIGS):

$$K = \begin{bmatrix} 13.17 & -7.79 & 0 \\ -7.79 & 15.57 & -7.79 \\ 0 & -7.79 & 7.79 \end{bmatrix} \times 10^6 \text{ n/m}$$

Effect Stiffness (BFGS):

$$K = \begin{bmatrix} 12.73 & -7.75 & 0.39 \\ -7.79 & 15.57 & -7.02 \\ 0.39 & -7.02 & 6.63 \end{bmatrix} \times 10^6 \text{ n/m}$$

Identified Stiffness:

$$K = \begin{bmatrix} 12.87 & -7.73 & 0.32 \\ -7.73 & 14.82 & 7.26 \\ 0.32 & 7.26 & 6.88 \end{bmatrix} \times 10^6 \text{ n/m}$$

FIGURE .I.

BIGGS' INFLEXIBLE GIRDER STRUCTURE - "BIGS"

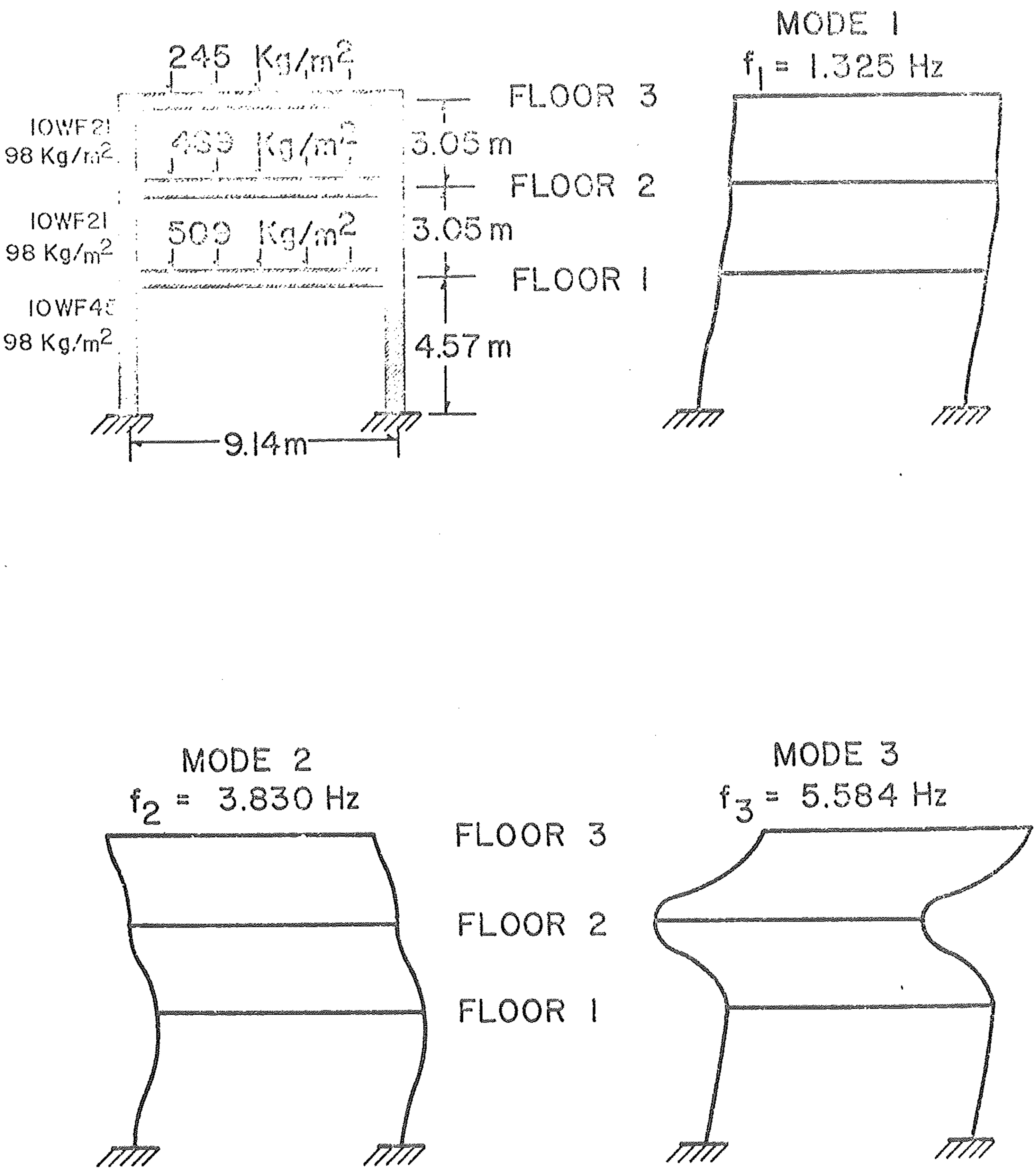


FIGURE .2.

BIGGS' FLEXIBLE GIRDER STRUCTURE - "BFGS"

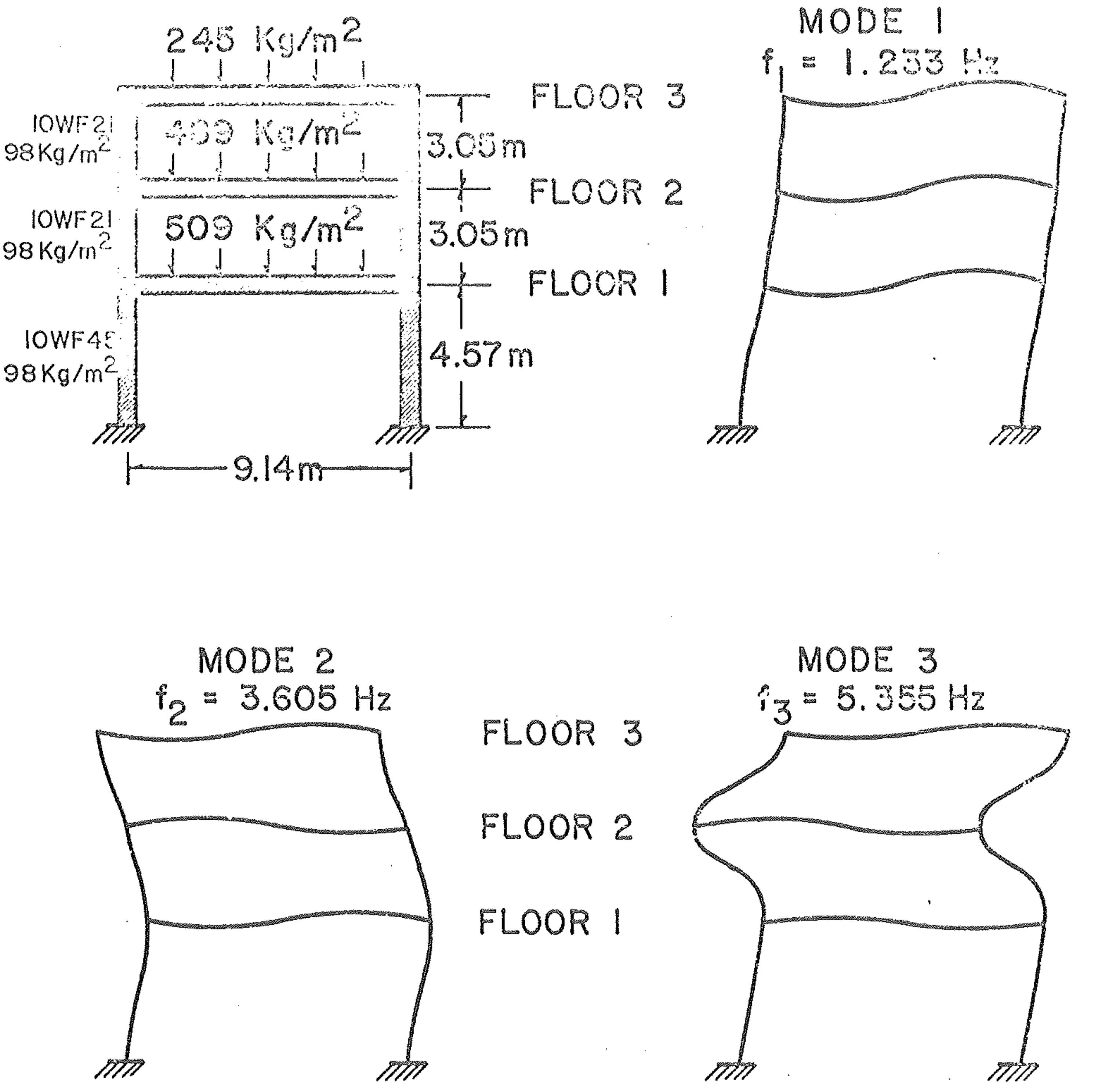
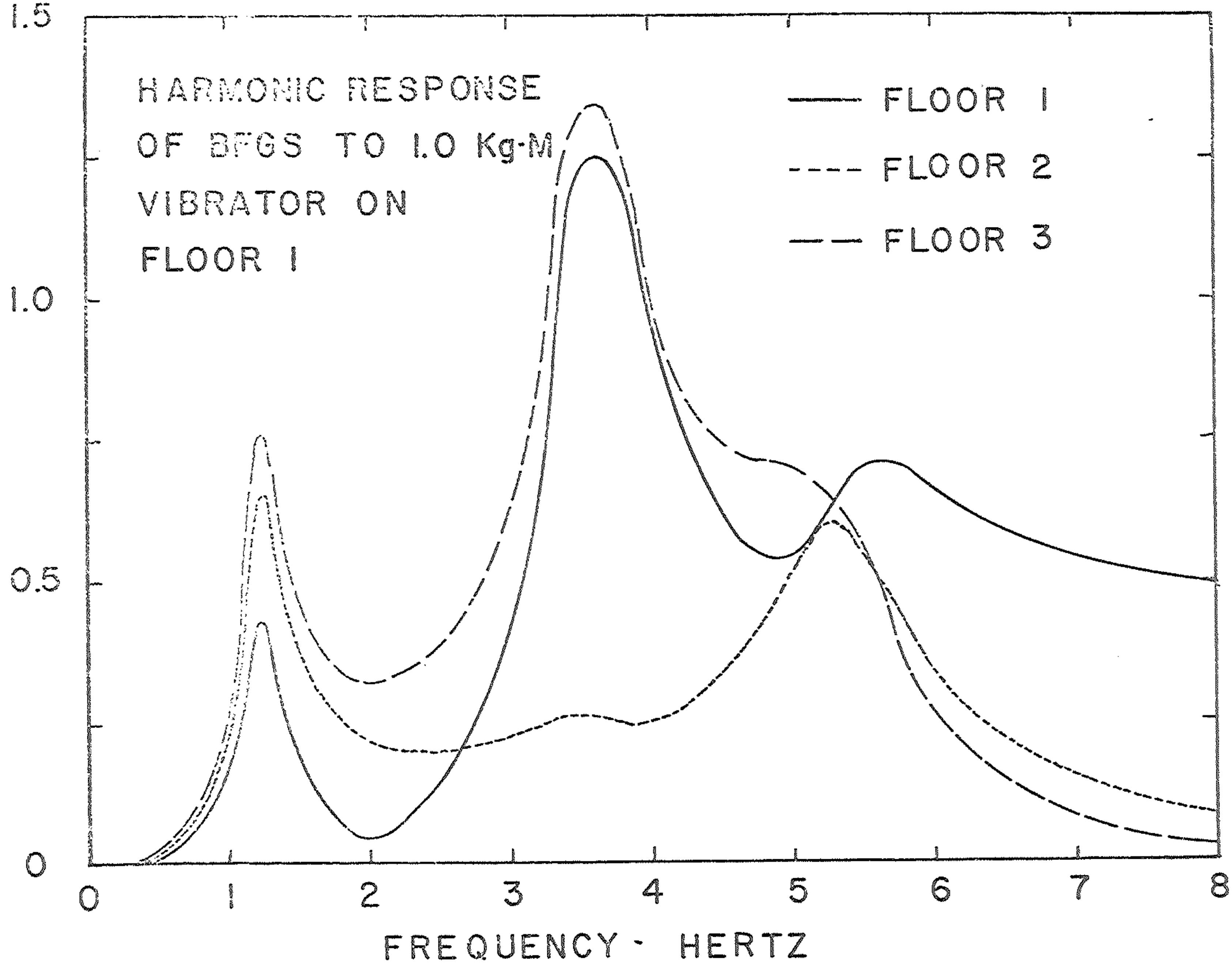


FIGURE .3.



1.5
1.0
0.5
0
HARMONIC RESPONSE
OF BFGS TO 1.0 Kg-M
VIBRATOR ON
FLOOR 1
FLOOR 1
FLOOR 2
FLOOR 3
0
1
2
3
4
5
6
7
8
FREQUENCY - HERTZ